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Michel Cristofol, Isma Kaddouri, Grégoire Nadin, Lionel Roques. Coefficient determination via asymptotic spreading speeds. *Inverse Problems*, 2014, 30 (3), pp.035005. 10.1088/0266-5611/30/3/035005 . hal-01072248

**HAL Id: hal-01072248**

**<https://hal.science/hal-01072248>**

Submitted on 7 Oct 2014

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# Coefficient determination via asymptotic spreading speeds

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**Abstract.** In this paper, we analyze the inverse problem of determining the reaction term  $f(x, u)$  in reaction-diffusion equations of the form  $\partial_t u - D\partial_{xx}u = f(x, u)$ , where  $f$  is assumed to be periodic with respect to  $x \in \mathbb{R}$ . Starting from a family of exponentially decaying initial conditions  $u_{0,\lambda}$ , we show that the solutions  $u_\lambda$  of this equation propagate with constant asymptotic spreading speeds  $w_\lambda$ . Our main result shows that the linearization of  $f$  around the steady state 0,  $\partial_u f(x, 0)$ , is uniquely determined (up to a symmetry) among a subset of piecewise linear functions, by the observation of the asymptotic spreading speeds  $w_\lambda$ .

## 1. Introduction

This paper is devoted to the reconstruction of the linear part  $\partial_u f(x, 0)$  of the reaction term  $f(x, u)$  in the following reaction-diffusion problem:

$$(P) \begin{cases} \partial_t u - D\partial_{xx}u = f(x, u), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

This equation describes the space-time evolution of a concentration  $u(t, x)$  in a heterogeneous excitable environment [4, 47]. It appears in several fields of applications such as physics, in combustion flame propagation models [9], in chemistry [11], in ecology, to study the dynamics of a population [36, 44] as well as in population genetics [2, 3, 40].

For Kolmogorov-Petrovsky-Piskunov type (KPP) nonlinearities (see definition below), it is well known that the solution  $u(t, x)$  of the problem (P) mainly depends on the reaction term  $f(x, u)$  through its linearization  $\partial_u f(x, 0)$  around the steady state 0 (see for exemple [7, 13]). In population dynamics, this term represents the intrinsic

growth rate of a population. The persistence or the extinction of the population and its rate of range expansion only depend on  $f(x, u)$  through this term  $\partial_u f(x, 0)$  [4, 7, 8, 22, 27]. In practice, this term is often unknown or partially known.

In this work, we assume that  $D > 0$ , and that  $x \mapsto f(x, u)$  is a 1-periodic reaction term, that is:

$$\forall x \in \mathbb{R}, \forall s \in \mathbb{R}, f(x, s) = f(x + 1, s). \quad (1)$$

The reconstruction of coefficients for parabolic equations is a widely studied problem. Several works are done in the case of parabolic operators defined on an open bounded set (see among others [1, 15, 20, 21, 30, 35, 37]). However in the unbounded case considered here, there are few references ([31, 33]) about coefficient identification.

In the case of bounded domains, several kinds of observations can be used for the determination of a coefficient. Many results are based on a method involving Carleman inequalities, which has been introduced in the seminal paper of Bukhgeim and Klibanov [12]. This method involves, in addition to localized observations of the solution  $u(t, x)$  of the considered problem, an observation of the solution on the whole domain of study, at some fixed time. Following the approach by Imanuvilov and Yamamoto [30] for the reconstruction of a potential in the linear case, several results have been obtained in the nonlinear case [17, 18, 19, 43]. More recently, other methods based on the maximum principle and the Hopf's lemma allowed to obtain uniqueness results using only pointwise observations in the one-dimensional case [16, 38, 39].

The study of the direct problem  $(P)$  has been widely developed recently (see e.g. [4, 5, 6, 7, 10, 45, 46]). The paper [5] is devoted to some nonlinear propagation phenomena in periodic and more general domains, for reaction-diffusion equations with KPP nonlinearities [34]. In [10], the authors establish spreading properties for the solutions of equations of the form  $\partial_t u - a(x)\partial_{xx}u - q(x)\partial_x u = f(x, u)$ . A common point of these works concerns the precise study of the speed of propagation of the solutions and of the asymptotic spreading speed.

On the other hand, the inverse problems associated with this class of equations have been little studied. In [14], the author treated the case of a linear parabolic operator in  $\mathbb{R}^n$ , associated with boundary and initial conditions which lead to a periodic solution. Then the problem stated in  $\mathbb{R}^n$  can be easily written into a problem stated in a bounded cell. The observations required to solve this inverse problem have to be measured on a set which includes all the edges of the cell which induces a strong constraint. In a recent work [31] the authors have improved this result by considering a nonlinear inverse parabolic problem with non-smooth coefficients and have obtained a more general reconstruction result without the above-mentioned constraint on the observation set.

Here, the initial condition  $u_0$  is not necessarily periodic, and the solution  $u(t, x)$  is therefore non-periodic in general. In this context, we use a new type of observations, which are based on the spreading properties of the solutions of the Cauchy problem  $(P)$ . Before going further on, we recall the classical Kolmogorov-Petrovsky-Piskunov

assumptions (denoted by KPP in reference to the seminal work [34]):

$$\begin{cases} \forall x \in \mathbb{R}, f(x, 0) = 0, \\ \exists M > 0, \text{ such that for } s \geq M, \text{ and } \forall x \in \mathbb{R}, f(x, s) \leq 0, \\ \forall x \in \mathbb{R}, s \mapsto \frac{f(x, s)}{s} \text{ is decreasing for } s > 0. \end{cases} \quad (2)$$

Under these hypotheses, the 1-periodicity assumption (1), and assuming that the steady state 0 is linearly unstable (see Proposition 2.1), it is known that the solution of the Cauchy problem (P), starting with a compactly supported or Heaviside initial condition, propagates (to the right) with a finite *asymptotic spreading speed*  $w^*$  [22] in the following sense:

$$\begin{cases} \lim_{t \rightarrow +\infty} u(t, x + ct) = 0, \forall c > w^*, x \in \mathbb{R}, \\ \liminf_{t \rightarrow +\infty} u(t, x + ct) > 0, \forall 0 \leq c < w^*, x \in \mathbb{R}. \end{cases} \quad (3)$$

This means that an observer moving to the right with a speed  $c$  larger than  $w^*$  will see the solution converge to 0, whereas if he moves with a speed smaller than  $w^*$ , he will see the solution remain above some positive constant. The existence of this constant asymptotic spreading speed, as well as the existence of traveling wave solutions which propagate with constant speeds  $c \geq w^*$  [4, 8, 45], are the key features which contributed to the success of the reaction-diffusion framework in applied sciences [36, 42, 44] since the pioneering work [34]. In particular, whenever the parameters  $D$ ,  $\partial_u f(x, 0)$  and  $u_0$  in (P) are properly fitted to experimental data, the predicted asymptotic spreading speed  $w^*$  is often in accordance with observations of species range expansions [42]. It is therefore natural to consider the inverse problem of coefficient determination in equation (P), based on observations of the spreading properties of its solution  $u(t, x)$ .

The main objective of this paper is to determine the linear part

$$r(x) = \partial_u f(x, 0) \quad (4)$$

of the reaction term  $f(x, u)$ , using observations of the spreading speed of the solution of the Cauchy problem (P). In that respect, instead of considering compactly supported or Heaviside initial conditions, we consider a 1-parameter family of initial conditions  $(u_{0,\lambda})_{\lambda \in I \subset (0, +\infty)}$ , which decay like  $e^{-\lambda x}$  as  $x \rightarrow +\infty$ , and we use the associated asymptotic spreading speeds  $w_\lambda$  as observations.

The outline of the paper is as follows. In Section 2, we give our assumptions and we state our main result, in Sections 3-5 we detail the proofs of our results.

## 2. Assumptions and main results

### 2.1. Assumptions on $f$

In addition to the periodicity assumption (1) and the KPP assumption (2), we assume that the nonlinearity  $(x, u) \mapsto f(x, u)$  is of class  $\mathcal{C}^{0,\alpha}$  with respect to  $x$  locally uniformly

in  $u \in \mathbb{R}$ , and of class  $\mathcal{C}^1$  with respect to  $u$ . Besides, we assume that the linear part of the reaction term,  $r(x) = \partial_u f(x, 0)$  belongs to the function space  $\mathcal{M}_1$  defined by:

$$\mathcal{M}_1 = \{1\text{-periodic function } \rho \in \mathcal{C}^0(\mathbb{R}), \text{ linear in } [0, \theta] \text{ and } [\theta, 1]\}, \quad (5)$$

for some (known) constant  $\theta \in (0, 1)$ . In other words,  $r(x)$  is a piecewise linear function.

## 2.2. Stationary states: existence, uniqueness and stability

Under our assumptions on the reaction term  $f$  and for continuous and bounded initial conditions  $u_0$ , it is known that the solution of the Cauchy problem (P) converges when  $t \rightarrow +\infty$  to a nonnegative stationary solution, that is, a solution  $p \geq 0$  of the problem:

$$-D \partial_{xx} p = f(x, p), \quad x \in \mathbb{R}. \quad (6)$$

More precisely, in [7] the authors have given a necessary and sufficient condition for the existence of a positive stationary state  $p$ . This condition is based on the description of the linear stability of the steady state 0, i.e., on the sign of the principal eigenvalue of the linear operator  $L_0$ :

$$L_0 : \psi \mapsto -D \partial_{xx} \psi - r(x) \psi, \quad (7)$$

with periodicity assumptions on  $\psi$ . We recall that, from the Krein-Rutman Theorem [24], there exists a unique  $k_0 \in \mathbb{R}$  and a unique function  $\psi_0 \in \mathcal{C}^2(\mathbb{R})$  such that

$$\begin{cases} -D \partial_{xx} \psi_0 - r(x) \psi_0 = k_0 \psi_0, & x \in \mathbb{R}, \\ \psi_0 \text{ is 1-periodic,} \\ \psi_0(x) > 0, & x \in \mathbb{R}, \\ \psi_0(0) = 1. \end{cases} \quad (8)$$

The results of [7] show that, under our assumptions on the function  $f$ ,

**Proposition 2.1.** *1) The equation (6) admits a positive and bounded solution  $p > 0$  if and only if  $k_0 < 0$ .*

*2) If  $k_0 < 0$ , the bounded solution  $p > 0$  of equation (6) is unique and 1-periodic in  $x$ .*

*3) Let  $u_0 \in \mathcal{C}^0(\mathbb{R})$  be bounded and such that  $u_0 \geq 0$ ,  $u_0 \not\equiv 0$ . If  $k_0 \geq 0$ , then the solution  $u(t, x)$  of (P) converges to 0 uniformly in  $x$  as  $t \rightarrow +\infty$ . If  $k_0 < 0$ , then  $u(t, x)$  converges to the solution  $p > 0$  of equation (6) locally uniformly in  $x$ , as  $t \rightarrow +\infty$ .*

The point 3) of the above proposition shows that, whatever the initial condition, the asymptotic spreading speed of the solution  $u(t, x)$  is equal to 0 if  $k_0 \geq 0$ . In the remaining part of the manuscript, we need to observe positive spreading speeds. As a consequence, we have to assume that  $k_0 < 0$ . A sufficient condition on  $r(x)$  such that  $k_0 < 0$  is (see proposition 2.9 in [7]):

$$\int_0^1 r(x) dx \geq 0, \text{ with } r \not\equiv 0 \text{ on } [0, 1]. \quad (9)$$

### 2.3. Initial conditions and asymptotic spreading speed

In the case of Heaviside and compactly supported initial conditions, the asymptotic spreading speed  $w^*$  exists, does not depend on this initial condition, is finite and satisfies (3).

Now, we consider a family of bounded front-like initial conditions  $u_{0,\lambda}$  satisfying for some  $\alpha \in (0, 1)$ ,  $x_0 > 0$  and all  $\lambda > 0$ :

$$u_{0,\lambda} \in \mathcal{C}^{0,\alpha}(\mathbb{R}), \quad u_{0,\lambda} \geq 0, \quad \liminf_{x \rightarrow -\infty} u_{0,\lambda}(x) > 0, \quad (10)$$

and

$$u_{0,\lambda}(x) = e^{-\lambda x} \text{ for } x \geq x_0. \quad (11)$$

We define the asymptotic spreading speed of the solution  $u_\lambda(t, x)$  of the Cauchy problem (P) starting with the initial condition  $u_0(x) = u_{0,\lambda}(x)$  as the nonnegative real number  $w_\lambda$  such that:

$$\begin{cases} \lim_{t \rightarrow +\infty} u_\lambda(t, x + ct) = 0, \quad \forall c > w_\lambda, x \in \mathbb{R}, \\ \liminf_{t \rightarrow +\infty} u_\lambda(t, x + ct) > 0, \quad \forall 0 \leq c < w_\lambda, x \in \mathbb{R}. \end{cases} \quad (12)$$

Depending on the initial condition, the asymptotic spreading speed could be finite or not [28, 41]. In the case of Heaviside and compactly supported initial conditions, we have already mentioned that the asymptotic spreading speed  $w^*$  is finite and (3) is satisfied. Besides, this asymptotic spreading speed is characterised by the following formula (see [22]):

$$w^* = \min_{\lambda > 0} \frac{-k_\lambda}{\lambda}, \quad (13)$$

where  $k_\lambda$  is the principal eigenvalue of the operator:

$$L_\lambda : \psi_\lambda \mapsto -D\psi_\lambda'' + 2\lambda D\psi_\lambda' - \lambda^2 D\psi_\lambda - r(x)\psi_\lambda, \quad (14)$$

with periodicity conditions, where  $r(x) = \partial_u f(x, 0)$ .

The next result describes the asymptotic spreading speed in terms of the initial condition. It is well known in the case of Heaviside and compactly supported initial conditions. To the best of our knowledge, it is not clearly stated in the existing literature for exponentially decaying initial conditions. In this proposition we show the existence of a finite asymptotic spreading speed  $w_\lambda$  and we give a formula for  $w_\lambda$ , under our assumptions on the initial conditions  $u_{0,\lambda}$ .

**Proposition 2.2.** *Assume that  $f$  satisfies (1), (2), the regularity assumptions of Section 2.1 and that  $k_0 < 0$  (see Section 2.2). Let  $u_{0,\lambda}$  satisfy (10), (11) and  $u_\lambda(t, x)$  be the solution of problem (P) with initial condition  $u_{0,\lambda}(x)$ . Then, we can associate to  $u_\lambda(t, x)$  a finite asymptotic spreading speed  $w_\lambda$  such that*

$$\begin{cases} w_\lambda = \frac{-k_\lambda}{\lambda} \text{ if } 0 < \lambda < \lambda^*, \\ w_\lambda = w^* \text{ otherwise,} \end{cases} \quad (15)$$

where  $k_\lambda$  is the principal eigenvalue of the operator defined by (14) and  $\lambda^* = \lambda^*(r)$  is the unique positive real number such that  $w^* = \frac{-k_{\lambda^*}}{\lambda^*}$ .

*Proof.* See the proof of this proposition in Section 4.

#### 2.4. Uniqueness result

Finally we state our main uniqueness result.

**Theorem 2.1.** *Let  $f$  (resp.  $\tilde{f}$ ) check (1), (2), the regularity assumptions of Section 2.1 and assume that  $k_0 < 0$  (resp.  $\tilde{k}_0 < 0$ ) and  $r(x) \in \mathcal{M}_1$  (resp.  $\partial_u \tilde{f}(x, 0) = \tilde{r}(x) \in \mathcal{M}_1$ ). Let  $u_\lambda(t, x)$  (resp.  $\tilde{u}_\lambda(t, x)$ ), be the solution of the Cauchy problem (P) (resp.  $(\tilde{P})$ ) associated to  $f$  and  $u_{0,\lambda}(x)$  (resp.  $\tilde{f}$  and  $u_{0,\lambda}(x)$ ). If we assume that the asymptotic spreading speeds  $w_\lambda$  (resp.  $\tilde{w}_\lambda$ ) associated to  $u_\lambda$  (resp.  $\tilde{u}_\lambda$ ) coincide on a continuum of values of  $\lambda$ , that is to say*

$$\exists \lambda_1 > 0 \text{ such that } \forall \lambda \in (0, \lambda_1), w_\lambda = \tilde{w}_\lambda,$$

then

$$\tilde{r}(x) = r(x) \text{ or } \tilde{r}(x) = r(-x + \theta).$$

This result shows that the linear part  $r(x) = \partial_u f(x, 0)$  of the reaction term  $f(x, u)$  in (P) is uniquely determined (up to a symmetry) by the knowledge of the asymptotic spreading speeds  $w_\lambda$ , for  $\lambda \in (0, \lambda_1)$ .

The proof is based on the equality of the first and the second moments of the functions  $r(x)$  and  $\tilde{r}(x)$  (see Lemma 3.2 and Lemma 3.3). The derivation of the equality of the higher-order moments would be more involved, but would enable one to extend the result of Theorem 2.1 to more general classes of functions than those in  $\mathcal{M}_1$ .

### 3. Proof of the main Theorem

Our goal is to reconstruct any potential  $r(x)$  element of  $\mathcal{M}_1$ , from observations of the asymptotic spreading speed which corresponds to a physical and measurable observation. Let  $w_\lambda(r)$  and  $w_\lambda(\tilde{r})$  be the asymptotic spreading speed associated to  $r$  and  $\tilde{r}$  respectively.

From the assumption of Theorem 2.1, we have  $w_\lambda(r) = w_\lambda(\tilde{r})$  for all  $\lambda \in (0, \lambda_1)$ . Thus, from Proposition 2.2,  $k_\lambda(r) = k_\lambda(\tilde{r})$  for all  $\lambda \in (0, \min(\lambda_1, \lambda^*(r), \lambda^*(\tilde{r})))$ .

**Lemma 3.1.** *For any  $r \in \mathcal{M}_1$ , the function :  $\lambda \mapsto k_\lambda(r)$  is analytic.*

This result follows from the analyticity of the coefficients in (14) and from the simplicity of  $k_\lambda$ , the principal eigenvalue of  $L_\lambda$  (see [32]).

The analyticity of the functions  $\lambda \mapsto k_\lambda(r)$  and  $\lambda \mapsto k_\lambda(\tilde{r})$  implies that

$$(H) : k_\lambda(r) = k_\lambda(\tilde{r}), \text{ for all } \lambda > 0.$$

We want to prove that the property (H) implies that:

$$\tilde{r}(x) = r(x) \text{ or } \tilde{r}(x) = r(-x + \theta).$$

For this, we consider  $r(x), \tilde{r}(x)$  two elements of  $\mathcal{M}_1$  such that

$$r(x) = ax + b \text{ if } x \in [0, \theta), r(x) = cx + d \text{ if } x \in [\theta, 1), \text{ for } a, b, c, d \in \mathbb{R},$$

and

$$\tilde{r}(x) = \tilde{a}x + \tilde{b} \text{ if } x \in [0, \theta), \tilde{r}(x) = \tilde{c}x + \tilde{d} \text{ if } x \in [\theta, 1), \text{ for } \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}.$$

Using the regularity and periodicity hypothesis of  $r(x)$ , we deduce

$$\begin{cases} r(x) = ax + b, \text{ for } x \in [0, \theta), \\ r(x) = -\left(\frac{a\theta}{1-\theta}\right)x + b + \frac{a\theta}{1-\theta}, \text{ if } x \in [\theta, 1). \end{cases} \quad (16)$$

Thus, from (16), there exist  $A, B \in \mathbb{R}$  such that

$$\tilde{r}(x) = Ar(x) + B, \forall x \in \mathbb{R}. \quad (17)$$

The proof will be developed in several parts. In a first step we prove the lemma:

**Lemma 3.2.** *For any  $r, \tilde{r} \in \mathcal{M}_1$  and verifying the hypothesis (H), we have:*

$$\int_0^1 r(x) dx = \int_0^1 \tilde{r}(x) dx = \bar{r}.$$

*Proof.* We consider the following problem:

$$\begin{cases} -D\psi_\lambda'' + 2\lambda D\psi_\lambda' - (\lambda^2 D + r(x))\psi_\lambda = k_\lambda(r)\psi_\lambda & \text{in } \mathbb{R}, \\ \psi_\lambda > 0, \\ \psi_\lambda \text{ 1-periodic}, \\ \psi_\lambda(0) = 1. \end{cases} \quad (18)$$

According to [5] (Theorem 2.1 p.8), we have as  $\lambda \rightarrow +\infty$

$$k_\lambda(r) + \lambda^2 D \rightarrow -\int_0^1 r(x) dx, \text{ and } k_\lambda(\tilde{r}) + \lambda^2 D \rightarrow -\int_0^1 \tilde{r}(x) dx. \quad (19)$$

Thus, using property (H) and (19) we deduce the result of Lemma 3.2 (see another proof of this result in Lemma 5.1).  $\square$

**Lemma 3.3.** *For any  $r, \tilde{r} \in \mathcal{M}_1$ , and verifying the property (H), we have:*

$$\int_0^1 (r(x) - \bar{r})^2 dx = \int_0^1 (\tilde{r}(x) - \bar{r})^2 dx.$$



*Proof.* See the proof of this Lemma in Section 5. □

From (17) we obtain

$$\int_0^1 \tilde{r}^2(x) dx = A^2 \int_0^1 r^2(x) dx + 2AB \int_0^1 r(x) dx + B^2,$$

by using Lemma 3.2 and Lemma 3.3 we get

$$(1 - A^2) \int_0^1 r^2(x) dx = B \left( (1 + A) \int_0^1 r(x) dx \right),$$

for  $A \neq -1$  we obtain

$$(1 - A) \int_0^1 r^2(x) dx = B \int_0^1 r(x) dx,$$

by using the fact that  $B = (1 - A) \int_0^1 r(x) dx$ , for  $A \neq 1$ , we conclude that

$$\int_0^1 r^2(x) dx = \left( \int_0^1 r(x) dx \right)^2. \quad (20)$$

From the Cauchy inequality, the equality (20) is true if and only if  $r$  is constant. Thanks to (17) we deduce that  $\tilde{r}$  is also constant, and according to Lemma 3.2 we obtain  $r = \tilde{r}$ . By symmetry we also deduce that if  $\tilde{r}$  is constant then  $r$  is constant so  $\tilde{r} = r$ .

Now we consider the case  $A = 1$ . We have

$$\int_0^1 \tilde{r}(x) dx = \int_0^1 r(x) dx + B.$$

By using the fact that  $\int_0^1 r(x) dx = \int_0^1 \tilde{r}(x) dx$ , we conclude that  $B = 0$ , so  $r(x) = \tilde{r}(x)$ . Finally, if  $A = -1$ , then  $\tilde{r}(x) = -r(x) + B$ , so  $\tilde{r}(x) = r(-x + \theta)$ . □

*Remark 1.* In the homogeneous case, i.e, if  $f$  does not depend on  $x$ , the uniqueness result of Theorem 2.1 is obvious. In such case, it is known [2, 34] that case  $w^* = 2\sqrt{rD}$ , thus  $w^* = \tilde{w}^*$  implies that  $r = \tilde{r}$ .

#### 4. Proof of Proposition 2.2

Recall that for the Cauchy problem  $(P)$  associated with compactly supported or Heaviside initial conditions, we can define an asymptotic speed of propagation (see formula (3)). In the case of exponentially decaying initial conditions of the type (10)-(11) we have to prove the existence of this speed.

The proof is based on the construction of appropriate sub and super-solutions for the system  $(P)$ . We are going to use the notion of pulsating traveling wave for the equation:

$$\partial_t u = D \partial_{xx} u + g_l(x, u),$$

with  $g_l(x, u) = \frac{1}{l}f(x, lu)$  for some  $l > 0$ . Note that,

$$\partial_u g_l(x, 0) = \partial_u f(x, 0), \forall l > 0.$$

In view to carry out sub-solutions and super-solutions of problem (P), we consider the following problem:

$$\begin{cases} -D\partial_{xx}p_l = g_l(x, p_l), & x \in \mathbb{R}, \\ p_l \text{ is 1-periodic,} \\ p_l > 0. \end{cases} \quad (21)$$

Now, we consider the solution  $u_\lambda(t, x)$  of (P) with initial condition  $u_{0,\lambda}(x)$ . We construct a super-solution for this problem. In that respect we first look for  $l$  such that the solution  $p_l(x)$  of (21) satisfies:

$$\min_{\mathbb{R}} p_l > \sup_{\mathbb{R}} u_{0,\lambda}. \quad (22)$$

Let us show that such a value of  $l$  exists. Let  $p(x) = p_1(x)$  be the unique positive solution of

$$-D\partial_{xx}p = f(x, p), x \in \mathbb{R}. \quad (23)$$

It is known, from Proposition 2.1, that  $p(x)$  is strictly positive and periodic. Thus, it exists  $\delta_1, \delta_2 > 0$  such that

$$0 < \delta_1 < p(x) < \delta_2 \text{ on } \mathbb{R}. \quad (24)$$

Recall that the function  $p_l$  satisfies:  $-D\partial_{xx}p_l = \frac{1}{l}f(x, lp_l)$  for  $x \in \mathbb{R}$ . Let us set  $q(x) = lp_l(x)$ . Then  $q$  verifies the following equation:

$$-D\partial_{xx}q = f(x, q).$$

By uniqueness (Proposition 2.1), we deduce  $q(x) = p(x)$ , and finally  $p_l(x) = \frac{p(x)}{l}$ . From (24) we obtain:

$$\frac{\delta_1}{l} < p_l(x) < \frac{\delta_2}{l}. \quad (25)$$

Let  $l_1 < 1$  be such that  $\frac{\delta_1}{l_1} > \sup_{\mathbb{R}} u_{0,\lambda}$ . Then we can write

$$\min_{\mathbb{R}} p_{l_1} > \sup_{\mathbb{R}} u_{0,\lambda}. \quad (26)$$

In a first step we consider  $\lambda' \in (0, \lambda)$  such that  $\lambda' < \lambda^*$  and let  $c' = \frac{-k_{\lambda'}}{\lambda'} > c^* = \frac{-k_{\lambda^*}}{\lambda^*}$ .

We consider  $U_{c'}^{l_1}(t, x)$ , a pulsating traveling wave solution of the problem:

$$\begin{cases} \partial_t U_{c'}^{l_1} = D\partial_{xx}U_{c'}^{l_1} + g_{l_1}(x, U_{c'}^{l_1}), & t \in \mathbb{R}, x \in \mathbb{R}, \\ \forall z \in \mathbb{Z}, \forall (t, x) \in \mathbb{R} \times \mathbb{R}, U_{c'}^{l_1}(t + \frac{z}{c'}, x) = U_{c'}^{l_1}(t, x - z), \\ \forall (t, x) \in \mathbb{R} \times \mathbb{R}, 0 \leq U_{c'}^{l_1}(t, x) \leq p_{l_1}(x), \\ \lim_{x \rightarrow -\infty} |U_{c'}^{l_1}(t, x) - p_{l_1}(x)| = 0 \text{ and } \lim_{x \rightarrow +\infty} U_{c'}^{l_1}(t, x) = 0, \end{cases} \quad (27)$$

where the above limits hold locally in  $t$ . The existence of this solution follows from [8] and its uniqueness (up to a translation) is proved in [29].

From [25], it is known that the asymptotic behavior of  $U_{c'}^{l_1}(t, x)$  is:

$$U_{c'}^{l_1}(t, x) \sim B e^{-\lambda_{c'}(x-c't)} \psi_{\lambda_{c'}}(x), \text{ as } x \rightarrow +\infty, \text{ for } B > 0, t \in \mathbb{R}, \quad (28)$$

where  $\lambda_{c'}$  is such that

$$\lambda_{c'} = \inf\{\lambda > 0 \text{ such that } k_\lambda + \lambda c' = 0\}, \quad (29)$$

and  $\psi_{\lambda_{c'}}(x)$  is the eigenfunction of the operator  $L_{\lambda_{c'}}$  (see (14)) associated to the principal eigenvalue  $k_{\lambda_{c'}}$ . From the convexity of the map  $\lambda \mapsto -k_\lambda$  (see [7]) and by definition of  $\lambda^*$  (see Proposition 2.2), the equation  $k_\lambda = -\lambda c'$  admits at most two roots,  $\lambda^- \leq \lambda^* \leq \lambda^+$ . Since  $\lambda' < \lambda^*$ , we get that  $\lambda' = \lambda^- = \lambda_{c'}$ .

Moreover, since  $l_1 < 1$  and  $\frac{f(x, s)}{s}$  is decreasing we have

$$\frac{f(x, l_1 s)}{l_1 s} > \frac{f(x, s)}{s}.$$

Thus  $g_{l_1} = \frac{1}{l_1} f(x, l_1 s) > f(x, s)$  for all  $s \geq 0$ .

Finally  $U_{c'}^{l_1}(t, x)$  satisfies

$$\partial_t U_{c'}^{l_1} - D \partial_{xx} U_{c'}^{l_1} = g_{l_1}(x, U_{c'}^{l_1}) > f(x, U_{c'}^{l_1}), x \in \mathbb{R}.$$

Then  $U_{c'}^{l_1}(t, x)$  is a super-solution for the problem (P).

Now we want to prove that

$$U_{c'}^{l_1}(0, x) \geq u_{0, \lambda}(x), \forall x \in \mathbb{R}. \quad (30)$$

From (28), we get

$$U_{c'}^{l_1}(0, x) \sim B e^{-\lambda' x} \psi_{\lambda'}(x), \text{ as } x \rightarrow +\infty.$$

Using that  $\psi_{\lambda'}(x)$  is continuous, 1-periodic,  $\psi_{\lambda'}(x)$  strictly positive on  $[0, 1]$  and  $\lambda' < \lambda$ , we obtain

$$B e^{-\lambda' x} \psi_{\lambda'}(x) > e^{-\lambda x}, \text{ for } x \text{ large enough}. \quad (31)$$

Then, there exists  $M > 0$  such that for any  $x > M$ ,

$$U_{c'}^{l_1}(0, x) \geq e^{-\lambda x} = u_{0, \lambda}(x). \quad (32)$$

From (26) and (27), we deduce that there exists  $m < 0$  such that

$$\min_{x \leq m} U_{c'}^{l_1}(0, x) \geq \sup_{x \in \mathbb{R}} u_{0, \lambda}(x). \quad (33)$$

It remains to prove that

$$\min_{x \in [m, M]} U_{c'}^{l_1}(0, x) \geq \sup_{x \in \mathbb{R}} u_{0, \lambda}(x).$$

Let

$$\delta = \min_{x \in [m, M]} U_{c'}^{l_1}(0, x). \quad (34)$$

From the strong parabolic maximum principle, we know that  $\delta > 0$ . Let  $L \geq m$  be large enough such that

$$e^{-\lambda x} < \delta, \text{ for all } x > L.$$

Then  $U_{c'}^{l_1}(0, x - L + m)$  verifies (30). Indeed:

- If  $x \leq L$ , then  $x - L + m \leq m$ . From (33), it follows that

$$U_{c'}^{l_1}(0, x - L + m) \geq \sup_{x \in \mathbb{R}} u_{0,\lambda}(x).$$

- If  $x \in (L, L + M - m)$ , then  $m \leq x - L + m \leq M$  and, from (34),

$$U_{c'}^{l_1}(0, x - L + m) > \delta > e^{-\lambda x} = u_{0,\lambda}(x).$$

- If  $x \geq L + M - m$ , then  $x - L + m \geq M$  and, from (32),

$$U_{c'}^{l_1}(0, x - L + m) \geq e^{-\lambda(x-L+m)} \geq e^{-\lambda x} e^{\lambda(L-m)} \geq e^{-\lambda x} = u_{0,\lambda}(x).$$

Thus, even if it means translating  $U_{c'}^{l_1}(0, x)$  to the right, we can assume that  $U_{c'}^{l_1}(0, x) > u_{0,\lambda}(x), \forall x \in \mathbb{R}$ . Since  $U_{c'}^{l_1}(0, x) > u_{0,\lambda}(x)$  and since  $U_{c'}^{l_1}$  is a super-solution of (P), a comparison principle implies that

$$\text{for all } \lambda' \in (0, \min(\lambda, \lambda^*)), U_{c'}^{l_1}(t, x) > u_{\lambda}(t, x), t \geq 0, x \in \mathbb{R}, \quad (35)$$

with  $c' = \frac{-k_{\lambda'}}{\lambda'}$ .

In a second step we consider  $\lambda''$  such that  $0 < \lambda < \lambda'' < \lambda^*$  and let  $c'' = \frac{-k_{\lambda''}}{\lambda''}$ . Let  $l_2 > 1$  be such that  $\frac{\delta_2}{l_2} < \liminf_{x \rightarrow -\infty} u_{0,\lambda}(x)$  (see (25)). We consider  $U_{c''}^{l_2}(t, x)$  the pulsating traveling wave solution of the problem (27) (with  $l_2$  instead of  $l_1$  and  $c''$  instead of  $c'$ ). Using the same arguments as above, and the fact that  $\max_{\mathbb{R}} p_{l_2} < \frac{\delta_2}{l_2} < \liminf_{x \rightarrow -\infty} u_{0,\lambda}(x)$ , it is easily seen that  $U_{c''}^{l_2}(t, x)$  is a sub-solution of the problem (P) satisfying

$$U_{c''}^{l_2}(0, x) < u_{0,\lambda}(x)$$

(even if it means translating  $U_{c''}^{l_2}(0, x)$  to the left). A comparison principle implies that

$$\text{for all } \lambda'' \in (\lambda, \lambda^*), U_{c''}^{l_2}(t, x) < u_{\lambda}(t, x), t \geq 0, x \in \mathbb{R}. \quad (36)$$

So we get from (35) and (36) an appropriate pair of sub and super-solutions for the problem (P), and we can write:

$$U_{c''}^{l_2}(t, x) < u_{\lambda}(t, x) < U_{c'}^{l_1}(t, x), \text{ for all } t \geq 0, x \in \mathbb{R}, \lambda \in (\lambda', \lambda'').$$

Now we are going to complete the proof of the existence of an asymptotic spreading speed associated to problem (P) with initial conditions of the type (10)-(11).

- *First case:* if  $\lambda < \lambda^*$ , we prove that  $w_{\lambda} = \frac{-k_{\lambda}}{\lambda}$  verifies (12).

- (i) Let  $c > w_\lambda = \frac{-k_\lambda}{\lambda}$ , we want to prove that

$$\lim_{t \rightarrow +\infty} u_\lambda(t, x + ct) = 0, \text{ for all } x \in \mathbb{R}.$$

Let  $\lambda' \in (0, \lambda)$  be such that  $\frac{-k_\lambda}{\lambda} < \frac{-k_{\lambda'}}{\lambda'} < c$ ; the existence of  $\lambda'$  follows from the continuity of  $s \mapsto \frac{k_s}{s}$ . We know from (35) that  $u_\lambda(t, x + ct) < U_{c'}^{l_1}(t, x + ct)$ , with  $w_\lambda < c' = \frac{-k_{\lambda'}}{\lambda'} < c$ . Since  $c' < c$  we obtain from (27)  $\lim_{t \rightarrow +\infty} U_{c'}^{l_1}(t, x + ct) = 0$ . We conclude that if  $c > w_\lambda$  then

$$\lim_{t \rightarrow +\infty} u_\lambda(t, x + ct) = 0, \text{ for all } x \in \mathbb{R}. \quad (37)$$

- (ii) Let  $c < w_\lambda = \frac{-k_\lambda}{\lambda}$ . We want to prove that

$$\liminf_{t \rightarrow +\infty} u_\lambda(t, x + ct) > 0.$$

Let  $\lambda'' \in (\lambda, \lambda^*)$  be such that  $c < \frac{-k_{\lambda''}}{\lambda''} < \frac{-k_\lambda}{\lambda}$ . We know from (36), that  $U_{c''}^{l_2}(t, x + ct) < u_\lambda(t, x + ct)$ , with  $c < c'' = \frac{-k_{\lambda''}}{\lambda''} < w_\lambda$ . Since  $c < c''$  we obtain from (27) that  $\liminf_{t \rightarrow +\infty} U_{c''}^{l_2}(t, x + ct) > 0$ . We conclude that if  $c < w_\lambda$  then

$$\liminf_{t \rightarrow +\infty} u_\lambda(t, x + ct) > 0, \text{ for all } x \in \mathbb{R}. \quad (38)$$

Using (37) together with (38) imply that  $w_\lambda = \frac{-k_\lambda}{\lambda}$  is the spreading speed of  $u_\lambda$  in the sense of (12) when  $\lambda \in (0, \lambda^*)$ .

- *Second case:* if  $\lambda \geq \lambda^*$ , we prove that  $w_\lambda = w^*$ .

- (i) Let  $c > w^*$ , and consider  $\lambda' \in (0, \lambda^*)$  such that  $w^* < \frac{-k_{\lambda'}}{\lambda'} = c' < c$ . From (35), we know that

$$0 < u_\lambda(t, x) < U_{c'}^{l_1}(t, x), \quad t \geq 0, x \in \mathbb{R}.$$

From (3) and since  $0 < c' < c$ , it follows that

$$\lim_{t \rightarrow +\infty} u_\lambda(t, x + ct) = 0, \text{ for all } x \in \mathbb{R}. \quad (39)$$

- (ii) Let  $c < w^*$  and  $V(t, x)$  be a solution of (P) with compactly supported initial condition such that  $0 \leq V(0, x) < u_{0, \lambda}(x)$  for all  $x \in \mathbb{R}$ . A comparison principle implies that

$$0 < V(t, x) < u_\lambda(t, x) \text{ for all } t > 0, x \in \mathbb{R}.$$

Thus

$$u_\lambda(t, x + ct) > V(t, x + ct), \text{ for all } t > 0, x \in \mathbb{R}.$$

From (27), we know that

$$\liminf_{t \rightarrow +\infty} V(t, x + ct) > 0, \text{ for all } x \in \mathbb{R}.$$

Thus

$$\liminf_{t \rightarrow +\infty} u_\lambda(t, x + ct) > 0, \text{ for all } x \in \mathbb{R}. \quad (40)$$

Using (39) together with (40) imply that  $w^*$  is the spreading speed of  $u_\lambda$  for all  $\lambda \geq \lambda^*$ .

□

### 5. Proof of Lemma 3.3

We consider the following problem

$$\begin{cases} D\psi_\lambda'' - 2\lambda D\psi_\lambda' + r(x)\psi_\lambda = -l_\lambda(r)\psi_\lambda & \text{in } \mathbb{R}, \\ \psi_\lambda > 0, \\ \psi_\lambda \text{ 1-periodic}, \\ \psi_\lambda(0) = 1, \end{cases} \quad (41)$$

where  $l_\lambda(r) = k_\lambda(r) + \lambda^2 D$ . We define the functions  $r_\lambda$  and  $h_\lambda$  by

$$l_\lambda(r) = -\bar{r} + \frac{r_\lambda}{\lambda^2},$$

and

$$\psi_\lambda(x) = 1 + \frac{f_1(x)}{\lambda} + \frac{h_\lambda(x)}{\lambda^2},$$

where

$$f_1(x) := \frac{1}{2D} \int_0^x (r(s) - \bar{r}) ds.$$

We derive from (41):

$$\begin{aligned} 0 &= (2Df_1' - r(x) - l_\lambda(r)) + \frac{1}{\lambda}(-Df_1'' + 2Dh_\lambda' - r(x)f_1 - l_\lambda(r)f_1) \\ &\quad + \frac{1}{\lambda^2}(-Dh_\lambda'' - r(x)h_\lambda - l_\lambda(r)h_\lambda) \\ &= (-l_\lambda(r) - \bar{r}) + \frac{1}{\lambda}(-Df_1'' + 2Dh_\lambda' - r(x)f_1 + \bar{r}f_1) \\ &\quad + \frac{1}{\lambda^2}(-Dh_\lambda'' - r(x)h_\lambda + \bar{r}h_\lambda) - \frac{1}{\lambda^3}r_\lambda f_1 - \frac{1}{\lambda^4}r_\lambda h_\lambda \\ &= \frac{1}{\lambda}(-Df_1'' + 2Dh_\lambda' - r(x)f_1 + \bar{r}f_1) \\ &\quad + \frac{1}{\lambda^2}(-Dh_\lambda'' - r(x)h_\lambda + \bar{r}h_\lambda - r_\lambda) - \frac{1}{\lambda^3}r_\lambda f_1 - \frac{1}{\lambda^4}r_\lambda h_\lambda. \end{aligned} \quad (42)$$

Now we prove that  $a_2 := \lim_{\lambda \rightarrow +\infty} r_\lambda$  and  $f_2(x) := \lim_{\lambda \rightarrow +\infty} h_\lambda(x)$  are well-defined by using Lemma 5.1.

**Lemma 5.1.** *There exists a constant  $C > 0$  such that*

$$|l_\lambda(r) + \bar{r}| \leq C/\lambda.$$

**Proof.** We know (see [7] for example) that

$$\begin{aligned} l_\lambda(r) &= \sup\{l \in \mathbb{R}, \quad \exists \psi \in W_{per}^{2,\infty}(\mathbb{R}), \quad \psi > 0 \text{ in } \mathbb{R}, \quad -L_\lambda \psi \geq l\psi \text{ a.e. in } \mathbb{R}\}, \\ &= \inf\{l \in \mathbb{R}, \quad \exists \psi \in W_{per}^{2,\infty}(\mathbb{R}), \quad \psi > 0 \text{ in } \mathbb{R}, \quad -L_\lambda \psi \leq l\psi \text{ a.e. in } \mathbb{R}\}. \end{aligned}$$

Using  $\psi(x) = 1 + \frac{f_1(x)}{\lambda}$  as a test-function, taking  $\lambda$  large enough so that  $1 + \frac{f_1(x)}{\lambda} > 0$  for all  $x$ , we get almost everywhere:

$$\begin{aligned} -L_\lambda \psi &= -\frac{Df_1''(x)}{\lambda} + \frac{2\lambda Df_1'(x)}{\lambda} - r(x) - \frac{r(x)f_1(x)}{\lambda}, \\ &= -\frac{r'(x)}{2\lambda} + r(x) - \bar{r} - r(x) - \frac{r(x)f_1(x)}{\lambda}, \\ &\geq -\frac{C_0}{\lambda} - \bar{r}, \end{aligned}$$

where  $C_0$  is a constant which only depends on  $\|r\|_\infty$  and  $\|r'\|_\infty$ . Take  $C$  large enough so that

$$C + \frac{Cf_1(x)}{\lambda} + \bar{r}f_1(x) \geq C_0.$$

We derive from this inequality that

$$\left(-\frac{C}{\lambda} - \bar{r}\right)\psi \leq -\frac{C_0}{\lambda} - \bar{r} \quad \text{over } \mathbb{R}.$$

Hence  $-L_\lambda \psi \geq \left(-\frac{C}{\lambda} - \bar{r}\right)\psi$  over  $\mathbb{R}$  and we derive from the definition of  $l_\lambda(r)$  that

$$l_\lambda(r) \geq -\frac{C}{\lambda} - \bar{r}.$$

Similarly, one can prove that, taking  $C$  larger if necessary,

$$l_\lambda(r) \leq \frac{C}{\lambda} - \bar{r}.$$

□

Define  $\theta_\lambda \in W_{per}^{2,\infty}(\mathbb{R})$  so that

$$\psi_\lambda(x) = 1 + \frac{f_1(x)}{\lambda} + \frac{\theta_\lambda(x)}{\lambda},$$

where

$$f_1(x) := \frac{1}{2D} \int_0^x (r(s) - \bar{r})ds.$$

Rewriting the identity  $-L_\lambda \psi_\lambda = l_\lambda(r)\psi_\lambda$  with this change of functions, we get:

$$-L_\lambda \theta_\lambda + \left(\bar{r} - \frac{r_\lambda}{\lambda^2}\right)\theta_\lambda = Df_1''(x) + (r(x) - \bar{r} + \frac{r_\lambda}{\lambda^2})f_1(x) + \frac{r_\lambda}{\lambda} =: F_\lambda. \quad (43)$$

Lemma 5.1 yields that  $\left|\frac{r_\lambda}{\lambda}\right| \leq C$  for all  $\lambda$ . Thus,  $F_\lambda \in L_{per}^\infty(\mathbb{R})$  and there exists two constants  $\tilde{\lambda} > 0, C_0 > 0$ , so that for all  $\lambda \geq \tilde{\lambda}$ ,  $\|F_\lambda\|_\infty \leq C_0$ .

Multiplying (43) by  $\theta_\lambda''$  and integrating, one gets, for all  $\lambda \geq \tilde{\lambda}$

$$\begin{aligned} \int_0^1 (\theta_\lambda'')^2 dx &= - \int_0^1 \left( r(x) - \bar{r} + \frac{r_\lambda}{\lambda^2} \right) \theta_\lambda \theta_\lambda'' dx - \int_0^1 F_\lambda \theta_\lambda'' dx, \\ &= \int_0^1 r'(x) \theta_\lambda \theta_\lambda' dx + \int_0^1 \left( r(x) - \bar{r} + \frac{r_\lambda}{\lambda^2} \right) (\theta_\lambda')^2 dx - \int_0^1 F_\lambda \theta_\lambda'' dx, \\ &\leq C_1 (\|\theta_\lambda'\|_{L^2(0,1)}^2 + \|\theta_\lambda''\|_{L^2(0,1)}), \end{aligned} \quad (44)$$

where  $C_1$  is a constant which only depends on  $\|r'\|_\infty, \|r\|_\infty$ , the constant  $C$  given by Lemma 5.1 and  $C_0$ , and where we have used the inequalities

$$\|\theta_\lambda\|_{L^2(0,1)} \leq \|\theta_\lambda\|_\infty \leq \|\theta_\lambda'\|_{L^1(0,1)} \leq \|\theta_\lambda'\|_{L^2(0,1)}, \quad (45)$$

since  $\theta_\lambda(0) = 0$ .

Assume now that there exists a sequence  $(\lambda_n)_n$  such that  $\lambda_n \geq \tilde{\lambda}$

and  $\|\theta_{\lambda_n}'\|_{L^2(0,1)} \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Let  $\zeta_n := \frac{\theta_{\lambda_n}}{\|\theta_{\lambda_n}'\|_{L^2(0,1)}}$ , so that  $\|\zeta_n'\|_{L^2(0,1)} = 1$ .

Dividing (44) by  $\|\theta_{\lambda_n}'\|_{L^2(0,1)}^2$ , one gets

$$\int_0^1 (\zeta_n'')^2 dx \leq C_1 \left( 1 + \frac{\|\zeta_n''\|_{L^2(0,1)}}{\|\theta_{\lambda_n}'\|_{L^2(0,1)}} \right).$$

Thus, as  $\lim_{n \rightarrow +\infty} \|\theta_{\lambda_n}'\|_{L^2(0,1)} = +\infty$ ,  $(\|\zeta_n''\|_{L^2(0,1)})_n$  is bounded. As  $\|\zeta_n'\|_{L^2(0,1)} = 1$  and  $\|\zeta_n\|_{L^2(0,1)} \leq 1$  for all  $n$  due to (45), we can extract a subsequence, that we still denote  $(\zeta_n)_n$ , which converges weakly in  $H_{per}^2(0,1)$  as  $n \rightarrow +\infty$ . On the other hand, equation (43) gives

$$2\lambda_n \zeta_n' = \zeta_n'' + \left( r(x) - \bar{r} + \frac{r_{\lambda_n}}{\lambda_n^2} \right) \zeta_n + \frac{F_{\lambda_n}}{\|\theta_{\lambda_n}'\|_{L^2(0,1)}},$$

and thus the sequence  $(\lambda_n \zeta_n')_n$  is bounded in  $L^2(0,1)$ , which is a contradiction since  $\|\zeta_n'\|_{L^2(0,1)} = 1$  for all  $n$ . We thus conclude that the family  $(\theta_\lambda')_{\lambda \geq \tilde{\lambda}}$  is bounded in  $L^2(0,1)$ .

It follows that  $(\theta_\lambda)_{\lambda \geq \tilde{\lambda}}$  and  $(\theta_\lambda'')_{\lambda \geq \tilde{\lambda}}$  are bounded in  $L^2(0,1)$  due to (44) and (45) respectively, and thus (43) implies that  $(\lambda \theta_\lambda')_{\lambda \geq \tilde{\lambda}}$  is bounded in  $L^2(0,1)$ . The same chain of inequalities yields that  $(\lambda \theta_\lambda)_{\lambda \geq \tilde{\lambda}}$  is bounded in  $H^2(0,1)$ .

By replacing in (43)  $\theta_\lambda$  by  $\frac{h_\lambda}{\lambda}$ , we get

$$0 = -Df_1'' + 2Dh_\lambda' + (\bar{r} - r)f_1 - \frac{1}{\lambda}(Dh_\lambda'' + (r - \bar{r})h_\lambda + r_\lambda) - \frac{r_\lambda f_1}{\lambda^2} - \frac{r_\lambda h_\lambda}{\lambda^3}. \quad (46)$$

Integrating and noticing that  $\int_0^1 (\bar{r} - r(x))f_1 dx = 2D \int_0^1 f_1 f_1' dx = D \int_0^1 (f_1^2)' dx = 0$ , one gets

$$0 = \frac{1}{\lambda} \int_0^1 (r(x) - \bar{r})h_\lambda dx + \frac{r_\lambda}{\lambda} + \frac{r_\lambda}{\lambda^2} \int_0^1 f_1 dx + \frac{r_\lambda}{\lambda^3} \int_0^1 h_\lambda dx$$



and thus

$$r_\lambda = - \int_0^1 (r(x) - \bar{r}) h_\lambda(x) dx - \frac{r_\lambda}{\lambda} \int_0^1 f_1 dx - \frac{r_\lambda}{\lambda^2} \int_0^1 h_\lambda dx. \quad (47)$$

We have shown previously that  $r_\lambda$  and  $\|h_\lambda\|_{H^2(0,1)}$  are bounded uniformly with respect to  $\lambda \geq \tilde{\lambda}$ . Then  $\frac{r_\lambda}{\lambda} \rightarrow 0$  as  $\lambda \rightarrow +\infty$ . Passing to the limit in (46), we get that  $f_2 := \lim_{\lambda \rightarrow +\infty} h_\lambda$  is well-defined in  $H^1(0,1)$  and that

$$f_2' = \frac{1}{2}(f_1'' + \frac{(r(x) - \bar{r})}{D} f_1).$$

Lastly, letting  $\lambda \rightarrow +\infty$  in (47), we obtain

$$\lim_{\lambda \rightarrow +\infty} r_\lambda = - \int_0^1 (r(x) - \bar{r}) f_2 dx = a_2.$$

Now, we can claim that  $a_2 := \lim_{\lambda \rightarrow +\infty} r_\lambda$  and  $f_2(x) := \lim_{\lambda \rightarrow +\infty} h_\lambda(x)$  are well-defined. By taking the limit in (46) we get:

$$-Df_1'' + 2Df_2' - r(x)f_1 + \bar{r}f_1 = 0, \quad (48)$$

and

$$a_2 = \int_0^1 (\bar{r} - r(x)) f_2(x) dx.$$

Define  $M(x) := \int_0^x r(y) dy$  and integrate by parts:

$$a_2 = - \int_0^1 (\bar{r}x - M(x)) f_2'(x) dx$$

(there is no boundary terms since  $M(1) = \int_0^1 r dx = \bar{r}$ ). We compute  $f_2'$  using the first equation in (48):

$$\begin{aligned} a_2 &= \frac{1}{2D} \int_0^1 (M(x) - \bar{r}x) (Df_1'' + (r(x) - \bar{r})f_1) dx, \\ &= -\frac{1}{2} \int_0^1 (r(x) - \bar{r}) f_1'(x) dx + \frac{1}{4D} \int_0^1 \frac{d}{dx} (M(x) - \bar{r}x)^2 f_1(x) dx, \\ &= -\frac{1}{2} \int_0^1 (r(x) - \bar{r}) f_1'(x) dx - \frac{1}{4D} \int_0^1 (M(x) - \bar{r}x)^2 f_1'(x) dx, \\ &= -\frac{1}{4D} \int_0^1 (r(x) - \bar{r})^2 dx - \frac{1}{24D^2} \int_0^1 \frac{d}{dx} (M(x) - \bar{r}x)^3 dx, \\ &= -\frac{1}{4D} \int_0^1 (r(x) - \bar{r})^2 dx. \end{aligned}$$

Hence, we conclude that

$$l_\lambda(r) = -\bar{r} - \frac{1}{4D\lambda^2} \int_0^1 (r(x) - \bar{r})^2 dx + o\left(\frac{1}{\lambda^2}\right),$$

from which the conclusion follows from Lemma 3.2 and  $l_\lambda(r) = l_\lambda(\tilde{r})$  for all  $\lambda > 0$ .  $\square$

## Acknowledgments

This work was partially supported by the franco-algerian agreement Tassili 11 MDU 834 (second author).

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